Bull. Sci. math. 125, 4 (2001) 311–340

POSITIVE SOLUTIONS OF AMBROSETTI–PRODI PROBLEMS INVOLVING THE CRITICAL SOBOLEV EXPONENT

BY

THIERRY AUBIN [a], WENZHI WANG [b,1]

[a] *UFR 920, Université Paris VI, 75252 Paris cedex 05, France*
[b] *Dept. of Math. Taiyuan Univ. of Technology, Taiyuan 030024, PR China*

Manuscript presented by T. AUBIN, received in October 2000

RÉSUMÉ. – Sur une variété Riemannienue compacte (M, g) de dimension $n \geqslant 3$, avec ou sans bord, nous étudions le problème d'Ambrosetti–Prodi avec exposant critique de Sobolev et $r(x) \geqslant 0$ $(r \not\equiv 0)$. Il s'agit de résoudre Eq. (0,1). Nous supposous $K > 0$ et $\Delta + a$ coercif. Il existe $\lambda^* > 0$ tel que Eq. (0,1) admet une solution (minimale) si et seulement si $0 < t \leqslant \lambda^*$. Pour $t = \lambda^*$ cette solution est unique, et il existe au moins deux solutions pour $0 < t < \lambda^*$. Certaines hypothèses supplémentaires sont requises suivant les valeurs de n. Le cas où $\Delta + a$ est le laplacien conforme est étudié. © 2001 Éditions scientifiques et médicales Elsevier SAS

Mots Clés: Elliptic equations; Compact manifolds; Critical Sobolev exponents

Let (M, g) be a C^∞ compact Riemannian manifold (with or without boundary) of dimension $n \geqslant 3$. Consider the following problem

$$(0.1)\qquad \begin{aligned} &\Delta u + au = Ku^{N-1} + tr(x) && \text{in } M,\\ &u > 0 && \text{in } M,\\ &u = 0 && \text{on } \partial M, \end{aligned}$$

[1] Partially supported by Foundations for Returned Overseas Chinese and by Natural Science Foundations of Shanxi Province.

where $\Delta = -\nabla^k\nabla_k$ is the Laplacian, $N = 2n/(n-2)$; a, K and r are some sufficiently smooth and nonidentically zero functions; and t is a positive real parameter.

This kind of problems is called the Ambrosetti–Prodi problem. A lot of literature is devoted to the study of such equations (see [11,14,20] and the references therein). In the past, considerations have been restricted to (i) bounded domains of $\mathbb{R}^n$, (ii) solutions rather than positive ones, (iii) cases where the nonlinear term is a lower order perturbation of u^{N-1} as u tends to infinity.

The exponent $p = (n+2)/(n-2)$ in Eq. (0.1) makes the problem much complicated. In fact, there is a sharp contrast between the cases $p = (n+2)/(n-2)$ and $p < (n+2)/(n-2)$.

Let $M = \Omega$ be a bounded starshaped domain in $\mathbb{R}^n$ endowed with the usual metric, then the Pohozaev inequality shows that the problem

$$\Delta u = u^p, \quad u > 0 \text{ in } \Omega, u = 0 \text{ on } \partial\Omega$$

has no solution at all if $p = N-1$ while this equation possesses a solution if $1 < p < N-1$ (see [15]).

Notice that $N = 2n/(n-2)$ is the limiting exponent for the Sobolev embedding

$$\mathring{H}_1(M) \to L^p(M).$$

This embedding is not compact if $p = N$. Consequently, the functional corresponding to (0.1),

$$\Phi(u) = \frac{1}{2}\int_M \{|\nabla u|^2 + au^2\} - \int_M \left\{\frac{1}{N}K|u|^N + ru\right\},$$

does not satisfy the Palais–Smale condition. Thus, there are serious difficulties if one tries to seek a solution, especially a nonminimal solution of (0.1) through variational approach.

We have a similar situation in solving the Yamabe problem, which is equivalent to the existence of a positive solution for the Yamabe equation

$$\Delta u + Su = \lambda u^{N-1} \quad \text{on } M,$$

where $\Delta + S$ is the conformal Laplacian and λ some real number. The functional corresponding to the Yamabe equation does not verify the Palais–Smale condition too.

There are a lot of such analytical and geometrical problems with a lack of compactness: (a) the existence of nonminimal solutions for H-systems; (b) the existence of minimal immersions of 2-sphere, etc. The study of problem (0.1) will help us to understand these phenomena.

The paper is organized as follows: in Section 1, we present some properties of minimal solutions of problem (0.1). In the sequel, we are looking for a second solution. To this end, in Sections 2 and 3, we discuss the existence of positive solutions of the problem

$$\Delta u + au = Ku^{N-1} + f(x,u),\, u|_{\partial M} = 0,$$

with $f(x,0) = 0$. Our main results are given in Section 4. In various cases, we prove that for a nonnegative, nonidentically zero C^1 function r, there exists a real number $\lambda^* > 0$ such that problem (0.1) has:

(i) no solution for $t > \lambda^*$;
(ii) a unique solution for $t = \lambda^*$, and
(iii) at least two distinct solutions for $0 < t < \lambda^*$.

When $3 \leqslant n \leqslant 5$ and (M, g) without boundary, or $n \geqslant 3$, $K \equiv 1$ and (M, g) conformally flat (with or without boundary), no additional condition is needed.

1. Minimal solutions

Let (M, g) be a $n \geqslant 2$ dimensional compact Riemannian manifold. In this section, we consider the following nonlinear elliptic equation

$$(1.1)\qquad \begin{array}{ll} \Delta u + au = h(x,u) + tr(x) & \text{in } M, \\ u > 0 & \text{in } M, \\ u = 0 & \text{on } \partial M, \end{array}$$

where $\Delta = -\nabla^k \nabla_k$ is the Laplacian, h and r are some nonnegative C^1 functions and t a positive parameter. Furthermore, we assume that

$h \in C^{l,\alpha}(\overline{M} \times \mathbb{R}_+)$ $(0 < \alpha < 1)$ satisfies the following hypothesis:

(h$_1$) $h(x, 0) = 0,\ h(x, u) \geqslant 0,\ \forall (x, u) \in \overline{M} \times \mathbb{R}_+$.

(h$_2$) $\exists\, \kappa > 0$ such that $h'_u(x, u) \geqslant -\kappa, \forall (x, u) \in \overline{M} \times \mathbb{R}_+$.

(h$_3$) $h(x, u)$ is convex with respect to u, strictly convex for a fixed $x \in \overline{M}$.

(h$_4$) The operator $\Delta + (a - h'_u(x, 0))$ is coercive on $\mathring{H}_1(M)$;

(h$_5$) $$\begin{cases} h(x, u) = K(x)u^p + g(x, u), \quad K(x) > 0 \text{ in } \overline{M}; \\ g(x, u) = \mathrm{o}(u^p), \quad g'_u(x, u) = \mathrm{o}(u^{p-1}) \\ \quad \text{uniformly as } u \to +\infty\ (1 < p \leqslant (n + 2)/(n - 2)). \end{cases}$$

By using the method of Amann [10], we can prove

THEOREM 1.1. – *Suppose that the operator* $\Delta + a$ *is coercive on* $\mathring{H}_1(M)$ *with* $a \in C^1(\overline{M})$ *and that* $h \in C^{1,\alpha}(\overline{M} \times \mathbb{R}_+)$ $(0 < \alpha < 1)$ *satisfies hypotheses* (h$_1$)–(h$_5$). *Then given* $r \in C^1(\overline{M}),\ 0 \leqslant r \not\equiv 0$, *there exists* $\lambda^* \in (0, +\infty)$ *such that*

(i) *for* $0 < t < \lambda^*$, *problem* (1.1) *has a minimal solution* $u(t)$, *the map* $u(\cdot) : (0, \lambda^*) \to C(\overline{M})$ *is strictly increasing, there is no solution to problem* (1.1) *for* $t > \lambda^*$;

(ii) *the minimal solution* $\bar{u} = u(t)$ *is stable, i.e., there exists a constant* $\mu_t > 0$ *such that*

$$\int_M \left[|\nabla \nu|^2 + \left(a - h_u(x, \bar{u})\right)\nu^2\right] \geqslant \mu_t \int_M \nu^2$$

for all $\nu \in \mathring{H}_1(M)$;

(iii) *for* $t = \lambda^*$, (1.1) *has exactly one solution*;

(iv) $\exists\, \lambda_* \in (0, \lambda^*)$ *such that* (1.1) *has at least two distinct solutions for every* $t \in (\lambda_*, \lambda^*)$.

Remark 1.1. – If we have $K(x) \geqslant 0$ instead of $K(x) > 0$, then (i) and (ii) still hold. For the proof of Theorem 1.1, see Wang [35] or Aubin and Wang [9].

2. Variational method

In the previous section, we got a minimal solution. In order to prove the existence of a second solution (in Section 4), we have to study the following problem

$$\begin{aligned} &\Delta u + au = Ku^{N-1} + f(x,u) && \text{in } M,\\ &u > 0 && \text{in } M,\\ &u = 0 && \text{on } \partial M, \end{aligned} \tag{2.1}$$

where $f(x,0) \equiv 0$. Since we consider positive solutions, the value of f for $u < 0$ has nothing to do. Hence set $f(x,u) = 0$, for $u < 0$.

For $u \in \mathring{H}_1(M)$, let

$$J(u) = \frac{1}{2}\int_M \{|\nabla u|^2 + au^2\} - \int_M \left\{\frac{K}{N}(u^+)^N + F(x,u^+)\right\},$$

where $F(x,u) = \int_0^u f(x,s)\,\mathrm{d}s$, then positive solutions of (2.1) corresponds to critical points of J. J does not verify the Palais–Smale condition.

2.1. The best Sobolev constant

LEMMA 2.1 (see Aubin [5], Hebey and Vaugon [23]). – *Let* (M,g) *be a* C^∞ *compact Riemannian manifold* (*without boundary or with a sufficiently smooth boundary*) *of dimension* $n \geqslant 3$. *Then there exists a constant* A *such that*

$$\|u\|_N^2 \leqslant K^2(n,2)\|\nabla u\|_2^2 + a\|u\|_2^2 \tag{2.2}$$

for all $u \in \mathring{H}_1(M)$.

Here $\|\cdot\|_p$ denotes the L_p norm. $K(n,2)$ is the smallest constant for which the inequality holds and is called the best Sobolev constant. $K(n,2)$ depends only on n. Let $\Lambda = \inf_{u\in H_1(\mathbb{R}^n)} \|\nabla u\|_N^2/\|u\|_N^2$ then $K^2(n,2) = 1/\Lambda$. We know that the infimum above is achieved by the function (see [4] or [15])

$$U(x) = \left(1+|x|^2\right)^{1-n/2}, \tag{2.3}$$

or by the functions

$$U_\varepsilon(x) = \left(\varepsilon + |x|^2\right)^{1-n/2}. \tag{2.4}$$

Thus $\Lambda = \|\nabla U\|_2^2 / \|U\|_N^2$.

2.2. A criterion

Now we give a criterion for problem (2.1) to have a solution. Suppose that $f(x,u)$ can be expressed as

$$f(x,u) = bu + g(x,u), \tag{2.5}$$

with $b \in C(\overline{M})$, $g \in C(\overline{M} \times \mathbb{R}_+)$ and

$$g(x,u) = \mathrm{o}(u) \quad \text{uniformly as } u \to 0, \tag{2.6}$$

$$g(x,u) = \mathrm{o}\left(u^{N-1}\right) \quad \text{uniformly as } u \to +\infty. \tag{2.7}$$

Moreover, suppose that the operator $(\Delta + a - b)$ is coercive on $\mathring{H}_1(M)$, that is to say, there exists a constant $\mu > 0$ such that

$$\int_M \left[|\nabla u|^2 + (a-b)u^2\right] \mathrm{d}V \geqslant \mu \int_M u^2 \,\mathrm{d}V, \quad \forall u \in \mathring{H}_1(M). \tag{2.8}$$

THEOREM 2.1. – *Suppose that the function $K(x)$ is strictly positive on $\overline{M}$ and that the operator $\Delta + a$ is coercive on $\mathring{H}_1(M)$. Let conditions* (2.5)–(2.8) *be satisfied. If there exists a function $v_0 \in \mathring{H}_1(M)$, $0 \not\equiv v_0 \geqslant 0$, such that*

$$\sup_{t \geqslant 0} J(tv_0) < \frac{1}{n}(\max K)^{1-n/2} \Lambda^{n/2}, \tag{2.9}$$

then problem (2.1) *has a positive solution.*

The proof of Theorem 2.1 depends on a variant *of the mountain pass lemma without the Palais–Smale condition* and inequality (2.2). Thanks to inequality (2.2), we can follow [15] step by step to complete the proof of the above theorem. See also [18].

In general, it is difficult to estimate the $\sup_{t \geqslant 0} J(tv)$ in (2.9). However, this is somewhat easier if the lower order perturbation $f(x,u)$ is positive near a maximum point of K.

Suppose that K achieves its maximum at a point $P \in M - \partial M$ and suppose that there exists a neighborhood B of P such that

$$f(x,u) > 0 \quad \forall (x,u) \in B \times \mathbb{R}_+. \tag{2.10}$$

Set, for $u \in \mathring{H}_1(M)$,

$$E(u) = \int_M \left(|\nabla u|^2 + au^2\right), \qquad X(u) = \left[\int_M K|u|^N\right]^{2/N}.$$

Then we have

$$J(tu) = \frac{t^2}{2}E(u) - \frac{t^N}{N}\left[X(u)\right]^{N/2} - \int_M F(x,tu).$$

Therefore, for a nonnegative function $v \in \mathring{H}_1(M)$ satisfying $\operatorname{supp} v \subset B$ and $\|v\| \neq 0$, we have $J(tv) \to -\infty$ as $t \to +\infty$. Thus $\sup_{t \geqslant 0} J(tv)$ is achieved at a point $t_v \geqslant 0$. In fact, $t_v > 0$ as a consequence of (2.10). By differentiation and evaluation at $t = t_v$, we get

$$t_v E(v) - t_v^{N-1}\left[X(v)\right]^{N/2} - \int_M f(x, t_v v)v = 0.$$

Noting that the integral above is nonnegative, it follows that

$$t_v \leqslant \left[E(v)\right]^{(n-2)/4} / \left[X(v)\right]^{n/4}. \tag{2.11}$$

Since the function $t \mapsto \frac{1}{2}t^2E(v) - \frac{1}{N}t^N[X(v)]^{N/2}$ is increasing for $t \leqslant [E(v)]^{(n-2)/4}/[X(v)]^{n/4}$, using (2.11), we have

$$\sup_{t \geqslant 0} J(tv) \leqslant \frac{1}{n}\left[E(v)/X(v)\right]^{n/2} - \int_M F(x, t_v v). \tag{2.12}$$

In conclusion, we have the following

PROPOSITION 2.1. – *Suppose that conditions* (2.5)–(2.8) *and* (2.10) *are satisfied, then* (2.12) *holds for all nonnegative function* $v \in \mathring{H}_1(M)$ *satisfying* $\operatorname{supp} v \subset B$ *and* $\|v\| \neq 0$.

2.3. Preliminary test function estimate

In order to satisfy (2.9), it is crucial to select an appropriate test function v_0. We can use the test function of Aubin [3] or the test function of Schoen [30], or the test function introduced below. We need the following lemma.

LEMMA 2.2 (see Lemma 1 of Aubin [3]). – *In a normal coordinate system* $(x_1, x_2, \ldots, x_n)$ *centered at* $P \in M - \partial M$*, the expansion of* $\sqrt{|g|}$ *in a neighborhood of* P *is*

$$\sqrt{|g|} = 1 - \frac{1}{6} R_{ij} x^i x^j + \mathrm{O}\bigl(|x|^3\bigr),$$

where $|g|$ *is the metric determinant of* (M, g)*, and the Ricci tensor is taken at* P*.*

Let $(x_1, x_2, \ldots, x_n)$ be a normal coordinate system centered at $P \in M - \partial M$. Given $\delta > 0$, we define a smooth radial cut off function $\alpha_\delta(x)$ on $\overline{M}$, with support in $B_{2\delta}$ such that $\alpha_\delta(x) = 1$ for $x \in B_\delta$, B_δ being the ball of radius δ centered at P. Then, for $\varepsilon \ll \delta < d$ (the injectivity radius), define

$$u_\varepsilon(x) = \alpha_\delta(x)\bigl(\varepsilon + |x|^2\bigr)^{1-n/2}. \tag{2.14}$$

LEMMA 2.3. – *Assume* K *achieves its maximum at* $P \in M - \partial M$*, then as* $t \to 0^+$*, we have the following estimates*

$$X(u_\varepsilon) = [\max K]^{2/N} \|U\|_N^2 \,\varepsilon^{1-n/2}\bigl[1 + \mathrm{O}(\varepsilon)\bigr], \tag{2.15}$$

$$\|\nabla u_\varepsilon\|_2^2 = \begin{cases} \|\nabla_E U\|_2^2\, \varepsilon^{1-n/2} + \mathrm{O}\bigl(\varepsilon^{2-n/2}\bigr), & \text{if } n \geqslant 5, \\ \|\nabla_E U\|_2^2\, \varepsilon^{-1} + \mathrm{O}(|\log \varepsilon|), & \text{if } n = 4, \\ \|\nabla_E U\|_2^2\, \varepsilon^{-1/2} + \mathrm{O}(1), & \text{if } n = 3, \end{cases} \tag{2.16}$$

$$\|u_\varepsilon\|_2^2 = \begin{cases} \mathrm{O}\bigl(\varepsilon^{2-n/2}\bigr), & \text{if } n \geqslant 5, \\ \mathrm{O}(\log|\varepsilon|), & \text{if } n = 4, \\ \mathrm{O}(1), & \text{if } n = 3, \end{cases} \tag{2.17}$$

thus,

$$\frac{E(u_\varepsilon)}{X(u_\varepsilon)} = \begin{cases} [\max K]^{-2/N} \Lambda + \mathrm{O}(\varepsilon), & n \geqslant 5, \\ [\max K]^{-2/N} \Lambda + \mathrm{O}(\varepsilon \log \varepsilon), & n = 4, \\ [\max K]^{-2/N} \Lambda + \mathrm{O}\bigl(\varepsilon^{1/2}\bigr), & n = 3. \end{cases} \tag{2.18}$$

Here ∇_E denotes the gradient operator under the standard Euclidian metric.

Proof: Verification of (2.15). – Without loss of generality, assume that $K(P) = \max K = 1$. Since K achieves its maximum at P, we have $K(x) = 1 + \mathrm{O}(|x|^2)$ for $x \in B_{2\delta}$. Using Lemma 2.2, we get

$$\sqrt{|g(x)|} = 1 + \mathrm{O}\big(|x|^2\big), \quad x \in B_{2\delta},$$

hence,

$$\begin{aligned}\int_M K u_\varepsilon^N \,\mathrm{d}V &= \int_{|x|\leqslant 2\delta} K\alpha^N \big(\varepsilon + |x|^2\big)^{-n} \sqrt{|g|}\,\mathrm{d}x \\ &= \int_{|x|\leqslant \delta} \big(\varepsilon + |x|^2\big)^{-n} \big[1 + \mathrm{O}\big(|x|^2\big)\big]\,\mathrm{d}x + \mathrm{O}(1).\end{aligned}$$

On the other hand,

$$\int_{|x|\leqslant \delta} \big(\varepsilon + |x|^2\big)^{-n}\,\mathrm{d}x = \|U\|_N^N \varepsilon^{-n/2} + \mathrm{O}(1),$$

and

$$\int_{|x|\leqslant \delta} \big(\varepsilon + |x|^2\big)^{-n} |x|^2\,\mathrm{d}x = C\varepsilon^{1-n/2} + \mathrm{O}(1),$$

where $C = \int_{\mathbb{R}^n} (1 + |x|^2)^{-n} |x|^2\,\mathrm{d}x$. Thus, we get (2.15).

Verification of (2.16). – Since u_ε is radial in B_δ, we have $|\nabla u_\varepsilon| = |\nabla_E u_\varepsilon|$ for $x \in B_\delta$ and

$$\nabla_E u_\varepsilon = \frac{\nabla_E \alpha(x)}{(\varepsilon + |x|^2)^{(n-2)/2}} - \frac{(n-2)\alpha(x)x}{(\varepsilon + |x|^2)^{n/2}}.$$

Since $\alpha(x) \equiv 1$ for $x \in B_\delta$, it follows that

$$\int_M |\nabla u_\varepsilon|^2\,\mathrm{d}V = (n-2)^2 \int_{|x|\leqslant \delta} \frac{|x|^2[1 + \mathrm{O}(|x|^2)]\,\mathrm{d}x}{(\varepsilon + |x|^2)^n} + \mathrm{O}(1), \tag{2.19}$$

and

$$\int\limits_{|x|\leqslant\delta} \frac{(n-2)^2|x|^2}{(\varepsilon+|x|^2)^n}\,\mathrm{d}x = \|\nabla_E U\|_2^2\varepsilon^{1-n/2} + \mathrm{O}(1). \tag{2.20}$$

When $n \geqslant 5$, we have

$$\int\limits_{|x|\leqslant\delta} |x|^4\big(\varepsilon+|x|^2\big)^{-n}\mathrm{d}x = K_1\varepsilon^{2-n/2} + \mathrm{O}(1), \tag{2.21}$$

where $K_1 = \int_{\mathbb{R}^n}(1+|x|^2)^{-n}|x|^4\,\mathrm{d}x$. Thus, (2.16) holds for $n \geqslant 5$.

When $n = 4$,

$$\int\limits_{|x|\leqslant\delta} \frac{|x|^4\,\mathrm{d}x}{(\varepsilon+|x|^2)^n} = \int\limits_0^\delta \frac{\omega_3 r^7\mathrm{d}r}{(\varepsilon+r^2)^4} = \frac{1}{2}\omega_3|\log\varepsilon| + \mathrm{O}(1), \tag{2.22}$$

where ω_3 is the area of S^3. From (2.21), (2.22) and (2.24), we deduce that (2.16) holds for $n = 4$.

When $n = 3$,

$$\int\limits_{|x|\leqslant\delta} |x|^4\big(\varepsilon+|x|^2\big)^{-n}\mathrm{d}x = \int\limits_0^\delta \omega_2 r^6\big(\varepsilon+r^2\big)^{-3}\mathrm{d}r \leqslant \omega_2\delta,$$

with ω_2 being the area of S^2. Hence, (2.16) holds also for $n = 3$.

Verification of (2.17). – Above all we have,

$$\|u_\varepsilon\|_2^2 = \int\limits_{|x|\leqslant\delta} u_\varepsilon^2\sqrt{|g|}\,\mathrm{d}x + \mathrm{O}(1).$$

When $n \geqslant 5$,

$$\int\limits_{|x|\leqslant\delta} u_\varepsilon^2\sqrt{|g|}\,\mathrm{d}x \leqslant C \int\limits_{|x|\leqslant\delta} \big(\varepsilon+|x|^2\big)^{2-n}\mathrm{d}x = K_2\varepsilon^{2-n/2} + \mathrm{O}(1),$$

where $C = \max\sqrt{|g|}$ and $K_2 = C\int_{\mathbb{R}^n}(1+|x|^2)^{2-n}\,\mathrm{d}x$. Thus, (2.17) is proved when $n \geqslant 5$.

When $n=4$, (2.17) is verified by the following inequality:

$$\int_{|x|\leqslant\delta} u_\varepsilon^2\sqrt{|g|}\,\mathrm{d}x \leqslant C\int_{|x|\leqslant\delta}\left(\varepsilon+|x|^2\right)^{-2}\mathrm{d}x$$
$$= C\omega_3\int_0^\delta\left(\varepsilon+r^2\right)^{-2}r^3\,\mathrm{d}r = \frac{1}{2}C\omega_3|\log\varepsilon| + \mathrm{O}(1).$$

When $n=3$, (2.17) is verified by the following inequality:

$$\int_M u_\varepsilon^2\sqrt{|g|}\,\mathrm{d}x = \int_{|x|\leqslant 2\delta} u_\varepsilon^2\sqrt{|g|}\,\mathrm{d}x \leqslant \int_{|x|\leqslant 2\delta} C\left(\varepsilon+|x|^2\right)^{-1}\mathrm{d}x$$
$$= C\omega_2\int_{|x|\leqslant 2\delta}\left(\varepsilon+r^2\right)^{-1}r^2\,\mathrm{d}r \leqslant 2C\omega_2\delta. \qquad \square$$

PROPOSITION 2.2. – *Suppose that the function K achieves its maximum at a point $P\in M-\partial M$. Let $\nu_\varepsilon = u_\varepsilon/\sqrt{X(u_\varepsilon)}$, u_ε being given in* (2.14) *in a normal coordinate system centered at P. Then under the conditions of Proposition* 2.1, *inequality* (2.12) *holds for $\nu=\nu_\varepsilon$. Moreover, as $\varepsilon\to 0^+$,*

$$t_\varepsilon \to [\max K]^{2/N(2-N)}\Lambda^{1/(N-2)} = \tau. \tag{2.23}$$

Proof. – We deduce directly from Proposition 2.1 that

$$\sup_{t\geqslant 0} J(t\nu_\varepsilon) \leqslant \frac{1}{n}\left[E(\nu_\varepsilon)/X(\nu_\varepsilon)\right]^{n/2} - \int_M F(x,t_\varepsilon\nu_\varepsilon)\,\mathrm{d}V. \tag{2.24}$$

Since $\|\nu_\varepsilon\| = \mathrm{o}(1)$, it follows from Lemma 2.3 that

$$E(\nu_\varepsilon) = E(u_\varepsilon)/X(u_\varepsilon) = [\max K]^{-2/N}\Lambda + \mathrm{o}(1). \tag{2.25}$$

Noticing $X(\nu_\varepsilon)=1$, we have

$$E_\varepsilon - t_\varepsilon^{N-2} - \frac{1}{t_\varepsilon}\int_M f(x,t_\varepsilon\nu_\varepsilon)\nu_\varepsilon\,\mathrm{d}V = 0,$$

where $E_\varepsilon = E(\nu_\varepsilon)$. By (2.25), $E_\varepsilon \to [\max K]^{-2/N}\Lambda$. To prove (2.23), it is sufficient to show that

$$\frac{1}{t_\varepsilon}\int_M f(x, t_\varepsilon \nu_\varepsilon)\nu_\varepsilon \,\mathrm{d}V \to 0. \tag{2.26}$$

It follows from (2.5), (2.6) and (2.7) that, for any $\varepsilon' > 0$, there exists a constant $C > 0$ such that

$$|f(x,u)| \leqslant \varepsilon'|u|^{N-1} + C|u|,$$

thus,

$$\frac{1}{t_\varepsilon}\int_M |f(x, t_\varepsilon \nu_\varepsilon)|\nu_\varepsilon \,\mathrm{d}V \leqslant \varepsilon' t_\varepsilon^{N-2}\|\nu_\varepsilon\|_N^N + C\|\nu_\varepsilon\|_2^2. \tag{2.27}$$

On the other hand, we have

$$\|\nu_\varepsilon\|_N^N \leqslant [\min K]^{-1}[X(\nu_\varepsilon)]^{N/2} = [\min K]^{-1}.$$

We know by (2.11) that $t_\varepsilon \leqslant E_\varepsilon^{(n-2)/4}$. Consequently, (2.27) implies (2.26). □

Consider a conformal change of metric, $\tilde{g} = \varphi^{4/(n-2)}g$, $\varphi > 0$ on $\overline{M}$. We know that (see Aubin [1])

$$\Delta\varphi + S\varphi = \widetilde{S}\varphi^{N-1} \quad \text{in } M, \tag{2.28}$$

where $S = (n-2)R/4(n-1)$ and $\widetilde{S} = (n-2)\widetilde{R}/4(n-1)$, R and $\widetilde{R}$ being scalar curvatures of (M,g) and $(M,\tilde{g})$ respectively. Hereafter, we will use $\widetilde{\Delta}$, $\widetilde{R}$ etc. to denote the gradient operator, the scalar curvature etc. for the conformal metric $\tilde{g}$. Under this conformal metric change, we have

$$\begin{aligned} &\mathrm{d}\widetilde{V} = \varphi^N \,\mathrm{d}V, \quad \text{and} \\ &|\widetilde{\nabla} u|^2 = \varphi^{2-N}|\nabla u|^2, \quad \forall u \in H_0^1(M). \end{aligned} \tag{2.29}$$

Thus

$$E(\varphi u) = \int_M \{|\widetilde{\nabla} u|^2 + [\widetilde{S} + \varphi^{2-N}(a-S)]u^2\}\,\mathrm{d}\widetilde{V} \equiv \widetilde{E}(u), \tag{2.30}$$

$$X(\varphi u) = \left[\int_M K u^N \, \mathrm{d}\widetilde{V} \right]^{2/N} \equiv \widetilde{X}(u). \tag{2.31}$$

Suppose that K achieves its maximum at a point $P \in M - \partial M$. Let $(x_1, x_2, \ldots, x_n)$ be a $\tilde{g}$-normal coordinate system centered at P. Set $\nu_\varepsilon = u_\varepsilon / \sqrt{\widetilde{X}(u_\varepsilon)}$, where u_ε is given by (2.14), then we have

PROPOSITION 2.3. – *Under the conditions of Proposition* 2.2*, for a strictly positive smooth function φ, we have*

$$\sup_{t \geqslant 0} J(t\varphi\nu_\varepsilon) \leqslant \frac{1}{n} \left[\widetilde{E}(\nu_\varepsilon) \right]^{n/2} - \int_M F(x, t_\varepsilon \varphi \nu_\varepsilon) \varphi^{-N} \mathrm{d}\widetilde{V}, \tag{2.32}$$

where t_ε is the maximum point of $J(t\varphi\nu_\varepsilon)$. Moreover, as $\varepsilon \to 0^+$,

$$t_\varepsilon \to [\max K]^{2/N(2-N)} \Lambda^{1/(N-2)} = \tau. \tag{2.33}$$

Proof. – Observe that

$$\int_M F(x, t\varphi\nu) \, \mathrm{d}V = \int_M F(x, t\varphi\nu) \varphi^{-N} \mathrm{d}\widetilde{V}.$$

Take $\delta > 0$ small enough, then (2.32) follows from Proposition 2.2. As for (2.33), since φ and φ^{-N} are bounded, we can prove it essentially as we did in the proof of Proposition 2.2. □

2.4. Regularity

Solutions given in this chapter lies in $\overset{\circ}{H}_1(M)$. In fact they belong to $L^\infty(M)$. That can be proven as in Trudinger [32]. By using the bootstrap method, we show that the smoothness is as high as K, f and ∂M permit. One can also obtain regularity as in [15] and [35] by using the results of [17].

3. Results for $\Delta u + au = K u^{N-1} + f(x, u)$, $f(x, 0) = 0$

In this section, we pursue the study done in Section 2. We find simple hypotheses, especially on K, under which condition (2.9) is satisfied.

Now return to the following problem

$$\begin{aligned}
&\Delta u + au = K u^{N-1} + f(x,u) && \text{in } M,\\
&u > 0 && \text{in } M,\\
&u = 0 && \text{on } \partial M,
\end{aligned} \tag{3.1}$$

where $f(x,0)=0$. We need following hypothesis

$$\text{(H)}\qquad \begin{cases}
(a)\ K \text{ is sufficiently smooth on } M, \text{ and } K(x) \geqslant K_0 > 0,\\
\qquad \forall x \in \overline{M},\\
(b)\ f \in C(\overline{M} \times \mathbb{R}_+) \text{ satisfying conditions (2.5)–(2.8)},\\
(c)\ \exists P \in \mathring{M} \text{ where } K \text{ achieves its maximum and } \exists B,\\
\qquad \text{a neighborhood of } P, \text{ such that } f(x,u) \geqslant 0,\ \forall x \in B\\
\qquad \text{and } \forall u \geqslant 0.
\end{cases}$$

3.1. The case where f is super-linear at infinity

THEOREM 3.1. – *Under hypothesis* (H)*, suppose that* f *is super linear at infinity, i.e.*

$$f(x,u)/u \xrightarrow{\text{uniformly}} +\infty, \quad \textit{in } x \in B$$

as $u \to +\infty$*. Then problem* (3.1) *has a solution when* $n \geqslant 4$*.*

Proof. – Let $(x_1, x_2, \ldots, x_n)$ be a normal coordinate system centered at P and $B_{2\delta}$ a ball of radius 2δ $(0 < 2\delta < d$, the injectivity radius). Take $\delta > 0$ small enough such that $B_{2\delta} \subset B$. Let $u_\varepsilon = u_{\delta,\varepsilon}$ be given in (2.14). Set $v_\varepsilon = u_\varepsilon/\sqrt{X(u_\varepsilon)}$ then $\operatorname{supp} v_\varepsilon \subseteq B_{2\delta}$. It follows from Proposition 2.2 that

$$\sup_{t\geqslant 0} J(tv_\varepsilon) \leqslant \frac{1}{n}\bigl[E(v_\varepsilon)\bigr]^{n/2} - \int_M F(x, t_\varepsilon v_\varepsilon)\,\mathrm{d}V, \tag{3.2}$$

where t_ε is the maximum point of $J(tv_\varepsilon)$ with $t_\varepsilon \to \tau > 0$. We deduce from Lemma 2.3 that

$$\begin{aligned}
\sup_{t\geqslant 0} J(tv_\varepsilon) \leqslant \frac{1}{n}[\max K]^{1-n/2}\Lambda^{n/2} - \int_M F(x, t_\varepsilon v_\varepsilon)\,\mathrm{d}V\\
+ \begin{cases} \mathrm{O}(\varepsilon), & \text{if } n \geqslant 5;\\ \mathrm{O}(\varepsilon \log \varepsilon), & \text{if } n = 4. \end{cases}
\end{aligned} \tag{3.3}$$

Before going on the concrete estimates, we need to show

$$T_\varepsilon \equiv \inf_{|x|\leqslant\varepsilon^{3/8}} F(x, t_\varepsilon v_\varepsilon)/v_\varepsilon^2 \to +\infty \quad (\varepsilon \to 0^+). \tag{3.4}$$

In fact, since $f(x,u)/u$ tends to infinity uniformly in $x \in B$ as $u \to +\infty$,

$$F(x,u)/u^2 \to +\infty, \quad \text{unniformly in } x \in B_\delta. \tag{3.5}$$

On the other hand, for $|x| \leqslant \varepsilon^{3/8}$, we have $u_\varepsilon^2 \geqslant 2^{-n}\varepsilon^{3(2-n)/4}$. We know by Lemma 2.3 that

$$X(u_\varepsilon) = K'\varepsilon^{(2-n)/2}\big(1 + \mathrm{O}(\varepsilon)\big) \tag{3.6}$$

for a constant $K' > 0$, thus for $|x| \leqslant \varepsilon^{3/8}$ we have

$$v_\varepsilon^2 = u_\varepsilon^2/X(u_\varepsilon) \geqslant C''\varepsilon^{(2-n)/4} \to +\infty,$$

for some constant $C'' > 0$. Since $t_\varepsilon \to \tau > 0$ as $\varepsilon \to 0^+$, (3.5) follows from the above inequality and (3.6).

Now, we estimate

$$\int_M F(x, t_\varepsilon v_\varepsilon)\,\mathrm{d}V \geqslant T_\varepsilon \int_{|x|\leqslant\varepsilon^{3/8}} t_\varepsilon^2 v_\varepsilon^2 \sqrt{|g|}\,\mathrm{d}x \geqslant CT_\varepsilon \int_{|x|\leqslant\varepsilon^{3/8}} v_\varepsilon^2\,\mathrm{d}x,$$

that is to say, there exists a constant $C > 0$, independent of ε, such that

$$\int_M F(x, t_\varepsilon v_\varepsilon)\,\mathrm{d}V \geqslant CT_\varepsilon[X(u_\varepsilon)]^{-1} \int_{|x|\leqslant\varepsilon^{3/8}} \big(\varepsilon + |x|^2\big)^{2-n}\,\mathrm{d}x. \tag{3.7}$$

When $n \geqslant 5$, we have

$$\int_{|x|\leqslant\varepsilon^{3/8}} \frac{\mathrm{d}x}{(\varepsilon + |x|^2)^{n-2}} = \int_{|x|\leqslant\varepsilon^{-1/8}} \frac{\varepsilon^{2-n/2}\,\mathrm{d}x}{(1 + |x|^2)^{n-2}} = K_1\big(1 + \mathrm{o}(1)\big)\varepsilon^{2-n/2}, \tag{3.8}$$

where $K_1 = \int_{\mathbb{R}^n}(1+|x|^2)^{2-n}\mathrm{d}x$. It follows from (3.6), (3.7) and (3.8) that

$$\int_M F(x, t_\varepsilon v_\varepsilon)\,\mathrm{d}V \geqslant C'T_\varepsilon t_\varepsilon^2 \varepsilon\big(1 + \mathrm{O}(\varepsilon)\big), \quad \text{if } n \geqslant 5, \tag{3.9}$$

where $C' > 0$ is a constant.

When $n=4$, we have

$$\int\limits_{|x|\leqslant\varepsilon^{3/8}} \frac{\mathrm{d}x}{(\varepsilon+|x|^2)^{n-2}} = \int\limits_0^{\varepsilon^{-1/8}} \frac{\omega\,\mathrm{d}x}{(1+|x|^2)^{n-2}} = \frac{\omega}{8}|\log\varepsilon| + \mathrm{O}(1), \tag{3.10}$$

where ω is the area of S^3.

It follows from (3.3), (3.4) and (3.7)–(3.11) that

$$\sup_{t\geqslant 0} J(t\nu_\varepsilon)$$
$$\leqslant \frac{1}{n}[\max K]^{1-n/2}\Lambda^{n/2} + \begin{cases} \mathrm{O}(\varepsilon) - T_{n,\varepsilon}\varepsilon, & \text{if } n\geqslant 5;\\ \mathrm{O}(\varepsilon\log\varepsilon) - T_{n,\varepsilon}\varepsilon|\log\varepsilon|, & \text{if } n=4,\end{cases}$$

where $T_{n,\varepsilon}\to+\infty$ as $\varepsilon\to 0^+$. Thus when $n\geqslant 4$, we have for $\varepsilon>0$ small enough,

$$\sup_{t\geqslant 0} J(t\nu_\varepsilon) < \frac{1}{n}[\max K]^{1-n/2}\Lambda^{n/2}.$$

This completes the proof of Theorem 3.1 according to Theorem 2.1. □

THEOREM 3.2. – *Let* $n=3$. *Suppose in addition to hypothesis* (H), *that* f *satisfies the following condition*

$$\lim_{u\to+\infty} f(x,u)/u^3 = +\infty \quad \textit{uniformly in } x\in B,$$

with a maximum point of K *in* B. *Then problem* (3.1) *has a solution.*

Proof. – In a normal coordinate system $(x_1, x_2, \ldots, x_n)$ centered at P, let $\nu_\varepsilon = u_\varepsilon/\sqrt{X(u_\varepsilon)}$ be the function as in the proof of Theorem 3.1. It follows from Proposition 2.2 and Lemma 2.3 that

$$\sup_{t\geqslant 0} J(t\nu_\varepsilon) \leqslant \frac{1}{3}[\max K]^{-1/2}\Lambda^{3/2} + \mathrm{O}\big(\varepsilon^{1/2}\big) - \int\limits_M F(x, t_\varepsilon\nu_\varepsilon)\,\mathrm{d}V, \tag{3.12}$$

where t_ε is the maximum point of $J(t\nu_\varepsilon)$, with $t_\varepsilon\to\tau>0$.

Since $\lim_{u\to+\infty} f(x,u)/u^3\to+\infty$ and $\nu_\varepsilon(x)\to+\infty$ uniformly in $|x|\leqslant\varepsilon^{3/8}$, we have, as $\varepsilon\to 0^+$,

$$T_\varepsilon \equiv \inf_{|x|\leqslant\varepsilon^{3/8}} F(x, t_\varepsilon\nu_\varepsilon)/t_\varepsilon^4\nu_\varepsilon^4 \to +\infty.$$

On the other hand

$$\begin{aligned}\int_M F(x, t_\varepsilon \nu_\varepsilon)\,\mathrm{d}V &\geqslant \int_{|x|\leqslant\varepsilon^{3/8}} F(x, t_\varepsilon \nu_\varepsilon)\sqrt{|g|}\,\mathrm{d}x\\ &\geqslant T_\varepsilon \int_{|x|\leqslant\varepsilon^{3/8}} \nu_\varepsilon^4\sqrt{|g|}\,\mathrm{d}x \geqslant C T_\varepsilon \int_{|x|\leqslant\varepsilon^{3/8}} \nu_\varepsilon^4\,\mathrm{d}x\\ &= \left[C T_\varepsilon / X^2(u_\varepsilon)\right] \int_{|x|\leqslant\varepsilon^{3/8}} \left(\varepsilon + |x|^2\right)^{-2}\mathrm{d}x,\end{aligned}$$

where C is the infimum of $\sqrt{|g|}$. At the same time, we know by Lemma 2.3 that

$$X(u_\varepsilon) = C'\varepsilon^{-1/2}\left(1 + \mathrm{O}(\varepsilon)\right),$$

where $C' = [\max K]^{-4/N}\|U\|_N^{-4}$. Moreover,

$$\int_{|x|\leqslant\varepsilon^{3/8}} \left(\varepsilon+|x|^2\right)^{-2}\mathrm{d}x = \varepsilon^{-1/2} \int_{|x|\leqslant\varepsilon^{-1/4}} \left(\varepsilon+|x|^2\right)^{-2}\mathrm{d}x = C''\varepsilon^{-1/2}\left(1+\mathrm{O}(1)\right),$$

where $C'' = \int_{\mathbb{R}^3}(1 + |x|^2)^{-2}\,\mathrm{d}x$. Thus for $\varepsilon > 0$ sufficiently small,

$$\int_M F(x, t_\varepsilon \nu_\varepsilon)\,\mathrm{d}V \geqslant K' T_\varepsilon \varepsilon^{1/2},$$

where $K' > 0$ is some constant independent of ε. Consequently, we have by (3.12)

$$\sup_{t\geqslant 0} J(t\nu_\varepsilon) \leqslant \frac{1}{3}[\max K]^{-1/2}\Lambda^{3/2} + \mathrm{O}\left(\varepsilon^{1/2}\right) - K' T_\varepsilon \varepsilon^{1/2}.$$

Thus, since $T_\varepsilon \to +\infty$, we have

$$\sup_{t\geqslant 0} J(t\nu_\varepsilon) < \frac{1}{3}[\max K]^{-1/2}\Lambda^{3/2}$$

for $\varepsilon > 0$ sufficiently small. So, applying Theorem 2.1 we get Theorem 3.2. □

THEOREM 3.3. – *Let $n \geqslant 4$ and let hypothesis* (H) *be satisfied. Then problem* (3.1) *has a solution provided that*

$$(a - S) + \frac{(n-2)(n-4)}{4(n-1)} \frac{\Delta K}{2K} < 0 \quad \text{at } P.$$

Here, $S = (n-2)R/4(n-1)$, R being the scalar curvature of (M, g).

Proof. – Let $\nu \in \mathring{H}_1(M)$, $0 \not\equiv \nu \geqslant 0$ and $\operatorname{supp} \nu \subset B$. We deduce from Proposition 2.1 that

$$\sup_{t \geqslant 0} J(t\nu) \leqslant \frac{1}{n} \left[\frac{E(\nu)}{X(\nu)} \right]^{n/2} - \int_M F(x, t_\nu \nu_\nu) \, \mathrm{d}V,$$

where t_ν is the maximum point of $J(t\nu)$. Since $F(x, \nu) \geqslant 0$ in $B \times \mathbb{R}_+$, we have

$$\sup_{t \geqslant 0} J(t\nu) \leqslant \frac{1}{n} \left[E(\nu)/X(\nu) \right]^{n/2}.$$

On the other hand, we know by Aubin [3] that there is a function $\nu \in \mathring{H}_1(M)$, $0 \not\equiv \nu \geqslant 0$ and $\operatorname{supp} \nu \subset B$, such that

$$E(\nu)/X(\nu) < [\max K]^{-2/N} \Lambda.$$

Thus,

$$\sup_{t \geqslant 0} J(t\nu) < \frac{1}{n} [\max K]^{(2-n)/2} \Lambda^{n/2}.$$

And this completes the proof of Theorem 3.3 by using Theorem 2.1. □

When $\Delta + a$ is the conformal Laplacian, i.e., $a = S$, we need more delicate estimates.

THEOREM 3.4. – *Define*

$$\Omega_1 = \{ Q \in M - \partial M \mid W(Q) \neq 0 \},$$
$$\Omega_2 = \{ Q \mid K(Q) = \max K \}.$$

Suppose that $\Delta + S$ is coercive on $\mathring{H}_1(M)$ and let hypothesis (H) *be satisfied with $P \in \Omega_1 \cap \Omega_2$. Then problem* (3.1) *has a solution if*

(i) $n = 6$ *and* $\Delta K(P) = 0$;
(ii) $n > 6$ *and* $\Delta\Delta K(P) = 0$.

Here, $W(P)$ is the Weyl tensor at P.

Proof. – As in the proof of Theorem 3.3, for a $\nu \in \mathring{H}_1(M)$, $0 \not\equiv \nu \geqslant 0$ and $\operatorname{supp} \nu \subset B$,

$$\sup_{t \geqslant 0} J(t\nu) \leqslant \frac{1}{n}\big[E(\nu)/X(\nu)\big]^{n/2} - \int_M F(x, t_\nu \nu_\nu)\,\mathrm{d}V.$$

We know by Aubin and Hebey [6] that, there is a function $\nu \in \mathring{H}_1(M)$, $0 \not\equiv \nu \geqslant 0$ such that

$$E(\nu)/X(\nu) < [\max K]^{-2/N}\Lambda.$$

Thus

$$\sup_{t \geqslant 0} J(t\nu) < \frac{1}{n}[\max K]^{(2-n)/2}\Lambda^{n/2}.$$

And the proof of Theorem 3.4 is complete by using Theorem 2.1.

Another proof of Theorems 3.3 and 3.4 can be accomplished as in Wang [35] by using the test function given in (2.14) in a conformal normal coordinate system. □

When (M, g) is locally conformally flat, the preceding theorem does not work since the Weyl tensor vanishes everywhere. In this case, the effect of the nonlinear term $f(x, u)$ appears to be more evident even if $f(x, u)$ is sub-linear as u tends to infinity.

THEOREM 3.5. – *Let* (M, g) *be a locally conformally flat manifold of dimension* $n \geqslant 3$ *and let hypothesis* (H) *be satisfied with* $a = S$. *If in a neighborhood* B *of* P, *a maximum point of* K, $f(x, u)$ *satisfies in addition*

$$\lim_{u \to \infty} \inf\big[F(x, u)/u^{N-1}\big] \geqslant C$$

uniformly in $x \in B$, *with* C *a positive constant, then problem* (3.1) *has a solution provided that*

$$\Delta^k K(P) = 0, \quad \forall\, 1 \leqslant k \leqslant (n-2)/4.$$

Here k *are integers and* $\Delta^k = \Delta\Delta^{k-1}$.

Proof. – Let $\tilde{g} = \varphi^{4/(n-2)} g$ be a conformal metric on M such that, in a $\tilde{g}$-normal coordinate system centered at P, $(x_1, x_2, \ldots, x_n)$, $\tilde{g}_{ij} = \delta_{ij}$. Thus the scalar curvature $\widetilde{R} = 0$, therefore $\widetilde{S} = 0$. In this system, let $u_\varepsilon(x) = u_{\varepsilon,\delta}(x)$ be the function given in (2.14). Set

$$\nu_\varepsilon(x) = u_\varepsilon(x) / \sqrt{\widetilde{X}\big(u_\varepsilon(x)\big)},$$

where

$$\widetilde{X}(u) = \left[\int_M K u^N(x)\, \mathrm{d}\widetilde{V}\right]^{2/N}.$$

Recall that

$$\widetilde{E}(u) = \int_M \big[|\nabla u|^2 + \widetilde{S} u^2\big]\, \mathrm{d}\widetilde{V}.$$

Choose $\delta > 0$ small enough such that $\operatorname{supp} \nu_\varepsilon(x) \subset B_{2\delta} \subset B$. Using Proposition 2.3, we obtain

$$\sup_{t \geqslant 0} J(t\varphi\nu_\varepsilon) \leqslant \frac{1}{n}\big[\widetilde{E}(\nu_\varepsilon(x))\big]^{n/2} - \int_M F(x, t_\varepsilon \nu_\varepsilon)\varphi^{-N}\, \mathrm{d}\widetilde{V},$$

where t_ε is the point where $J(t_\varepsilon \nu_\varepsilon)$ achieves its maximum. We saw that $t_\varepsilon \to \tau > 0$ as $\varepsilon \to 0$.

Now, let us prove that there exists a constant $c > 0$ such that, as $\varepsilon \to 0^+$,

$$\sup_{t \geqslant 0} J(t\varphi\nu_\varepsilon) \leqslant \frac{1}{n}[\max K]^{(2-n)/2} \Lambda^{n/2} - c\varepsilon^{(n-2)/4} + \mathrm{o}\big(\varepsilon^{(n-2)/4}\big). \tag{3.14}$$

Thus, as $\varepsilon \to 0$, we will have

$$\sup_{t \geqslant 0} J(t\varphi\nu_\varepsilon) < \frac{1}{n}[\max K]^{(2-n)/2} \Lambda^{n/2}.$$

And Theorem 2.1 achieves the proof.

In fact, we have the following estimates that we will verify later:

$$\widetilde{E}(u_\varepsilon) = \|\nabla_E U\|_2^2 \varepsilon^{(2-n)/2} + \mathrm{O}(1), \tag{3.15}$$

$$\widetilde{X}(u_\varepsilon) = [\max K]^{2/N} \|U\|_N^2 \varepsilon^{(2-n)/2}\big(1 + \mathrm{o}\big(\varepsilon^{(2-n)/4}\big)\big), \tag{3.16}$$

$$\int_M F(x, t_\varepsilon \varphi v_\varepsilon)\varphi^{-N}\,\mathrm{d}\widetilde{V} > c\varepsilon^{(2-n)/4}, \quad c > 0. \tag{3.17}$$

Thus

$$\widetilde{E}(v_\varepsilon) = \widetilde{E}(u_\varepsilon)/\widetilde{X}(u_\varepsilon) = [\max K]^{-2/N} \Lambda\big(1 + \mathrm{o}\big(\varepsilon^{(n-2)/4}\big)\big). \tag{3.18}$$

And (3.14) follows.

Verification of (3.15). – We have

$$\begin{aligned}
\widetilde{E}(u_\varepsilon) &= \int_M \big(|\nabla_{\tilde{g}} u_\varepsilon|^2 + \widetilde{S} u_\varepsilon^2\big)\,\mathrm{d}\widetilde{V} \\
&= \int_{|x| \leqslant 2\delta} |\nabla_E u_\varepsilon|^2\,\mathrm{d}\widetilde{V} = \int_{|x| \leqslant \delta} |\nabla_E u_\varepsilon|^2\,\mathrm{d}x + \mathrm{O}(1) \\
&= (n-2)^2 \int_{|x| \leqslant \delta} |x|^2 \big(\varepsilon + |x|^2\big)^{-n} \mathrm{d}x + \mathrm{O}(1) \\
&= (n-2)^2 \varepsilon^{(2-n)/2} \int_{\mathbb{R}^n} |x|^2 \big(1 + |x|^2\big)^{-n} \mathrm{d}x + \mathrm{O}(1) \\
&= \|\nabla_E U\|_2^2 \varepsilon^{(2-n)/2} + \mathrm{O}(1).
\end{aligned}$$

Verification of (3.16). – Set $l = (n-2)/2$. Since P is a maximum point of K, $\Delta^k K(P) = 0$ for $1 \leqslant k \leqslant l/2$ implies that $\nabla_\alpha K(P) = 0$, $\forall\ 1 \leqslant |\alpha| \leqslant l$. Thus, near P, K can be expressed as

$$K(x) = K(P) + \beta(x), \quad \beta(x) = \mathrm{o}\big(|x|^l\big).$$

Thus

$$\begin{aligned}
\int_M K u_\varepsilon^N(x)\,\mathrm{d}\widetilde{V} &= \int_{|x| \leqslant 2\delta} K u_\varepsilon^N(x)\,\mathrm{d}x \\
&= \int_{|x| \leqslant \delta} \big(K(P) + \beta(x)\big)\big(\varepsilon + |x|^2\big)^{-n} \mathrm{d}x + \mathrm{O}(1).
\end{aligned}$$

We know that

$$\int_{|x|\leqslant\delta} \big(\varepsilon+|x|^2\big)^{-n}\mathrm{d}x = \|U\|_N^N \varepsilon^{-n/2} + \mathrm{O}(1).$$

Let C and C_ε be the maxima of $|\beta(x)|/|x|^l$ in $|x|\leqslant\delta$ and in $|x|\leqslant\varepsilon^{1/4}$ respectively. Then

$$\begin{aligned}
&\int_{|x|\leqslant\delta} |\beta(x)|\big(\varepsilon+|x|^2\big)^{-n}\mathrm{d}x\\
&\quad\leqslant \int_{|x|\leqslant\varepsilon^{1/4}} |\beta(x)|\big(\varepsilon+|x|^2\big)^{-n}\mathrm{d}x + \int_{\varepsilon^{1/4}\leqslant|x|\leqslant\delta} |\beta(x)|\big(\varepsilon+|x|^2\big)^{-n}\mathrm{d}x\\
&\quad\leqslant C_\varepsilon \int_{|x|\leqslant\delta} |x|^l\big(\varepsilon+|x|^2\big)^{-n}\mathrm{d}x + C\int_{|x|\geqslant\varepsilon^{1/4}} |x|^l\big(\varepsilon+|x|^2\big)^{-n}\mathrm{d}x\\
&\quad\leqslant C_\varepsilon\varepsilon^{(l-n)/2}\int_{|x|\leqslant\delta} |x|^l\big(1+|x|^2\big)^{-n}\mathrm{d}x\\
&\qquad + C\varepsilon^{(l-n)/2}\int_{|x|\geqslant\varepsilon^{1/4}} |x|^l\big(1+|x|^2\big)^{-n}\mathrm{d}x = \mathrm{o}\big(\varepsilon^{(l-n)/2}\big).
\end{aligned}$$

Notice that C_ε and $\int_{|x|\geqslant\varepsilon^{-1/4}} |x|^l(1+|x|^2)^{-n}\,\mathrm{d}x$ tend to zero as ε tends to zero. Thus

$$\int_M K u_\varepsilon^N\,\mathrm{d}\widetilde{V} = K(P)\|U\|_N^N\varepsilon^{-n/2}\big(1+\mathrm{o}\big(\varepsilon^{l/2}\big)\big),$$

and we get (3.16) as expected.

Verification of (3.17). – It is known that $\nu_\varepsilon^2(x)\geqslant C\varepsilon^{(2-n)/2}\to\infty$ in $|x|\leqslant\varepsilon^{3/8}$. On the other hand, it follows from the conditions of the theorem, that

$$F(x,t_\varepsilon\varphi\nu_\varepsilon)\geqslant C\nu_\varepsilon^{N-1},\quad \forall|x|\leqslant\varepsilon^{3/8},$$

and

$$\int_M F(x,t_\varepsilon\varphi\nu_\varepsilon)\varphi^{-N}\,\mathrm{d}\widetilde{V}\geqslant C\int_{|x|\leqslant\varepsilon^{3/8}} \nu_\varepsilon^{N-1}\,\mathrm{d}x$$

$$= C[\widetilde{X}(u_\varepsilon)]^{(1-N)/2} \int\limits_{|x|\leqslant\varepsilon^{3/8}} u_\varepsilon^{N-1}\,\mathrm{d}x.$$

But

$$\int\limits_{|x|\leqslant\varepsilon^{3/8}} u_\varepsilon^{N-1}\,\mathrm{d}x = \int\limits_{|x|\leqslant\varepsilon^{3/8}} \left(\varepsilon+|x|^2\right)^{-(n+2)/2}\mathrm{d}x$$
$$= \varepsilon^{-1} \int\limits_{|x|\leqslant\varepsilon^{3/8}} \left(1+|x|^2\right)^{-(n+2)/2}\mathrm{d}x \geqslant C\varepsilon^{-1}.$$

We deduce from (3.16) that

$$[\widetilde{X}(u_\varepsilon)]^{(1-N)/2} \geqslant C\varepsilon^{(n+2)/4}.$$

Hence, we get (3.17) as expected. □

4. Multiple solutions

In this section, for simplicity, we will restrict ourselves to the following problem

$$(4.1)\qquad \begin{array}{ll} \Delta u + au = KU^{N-1} + tr(x) & \text{in } M,\\ u > 0 & \text{in } M,\\ u = 0 & \text{on } \partial M. \end{array}$$

Here $K(x)$ is a strictly positive function and $r(x)$ a nonnegative function on $\overline{M}$. We assume that the operator $\Delta + a$ is coercive on $\mathring{H}_1(M)$.

Set $h(x,u) = K(x)u^{N-1}$, then $h(x,u)$ meets all requirements of Theorem 1.1. Thus, there exists a $\lambda^* > 0$ such that problem (4.1) has:

(i) no solution for $t > \lambda^*$;
(ii) exactly one solution for $t = \lambda^*$;
(iii) for $0 < t < \lambda^*$, a minimal solution $\bar{u} = u(t)$ satisfying, for all $\nu \in \mathring{H}_1(M)$,

$$(4.2)\qquad \int\limits_M \left[|\nabla\nu|^2 + (a - h_u(x,\bar{u})\nu^2\right] \geqslant C\int\limits_M \nu^2.$$

In what follows, we will seek another solution.

For a fixed $t \in (0, \lambda^*)$, we will denote by $\bar{u} = u(t)$ the minimal solution of $(4.1)_t$. Since $\bar{u}$ is minimal, a second solution should have the form

$$\tilde{u} = \bar{u} + u, \quad u > 0 \text{ in } M, u|_{\partial M} = 0.$$

Thus, u satisfies the following equation

$$\Delta u + au = K\left[(u + \bar{u})^{N-1} - \bar{u}^{N-1}\right],$$

In other words, we are looking for a function u, such that

$$\begin{aligned} &\Delta u + au = Ku^{N-1} + f(x,u) && \text{in } M, \\ &u > 0 && \text{in } M, \\ &u = 0 && \text{on } \partial M, \end{aligned} \tag{4.3}$$

where

$$f(x,u) = K\left[(u + \bar{u})^{N-1} - u^{N-1} - \bar{u}^{N-1}\right]. \tag{4.4}$$

Obviously, f satisfies conditions (2.5)–(2.7). We deduce from (4.2) that condition (2.8) is satisfied with

$$b = h'_u(x, \bar{u}) = (N-1)K\bar{u}^{N-2}.$$

From expression (4.4), it is easily checked that

$$f(x,u) \geqslant 0, \quad \forall u \geqslant 0,\ \forall x \in M.$$

Thus Eq. (4.3) satisfies hypothesis (H) of Section 3. We will not repeat this fact later.

Let $P \in M - \partial M$ be a maximum point of K and let $B \Subset M$ be a neighborhood of P, then by (4.4) we have

$$\lim_{u\to\infty} \frac{f(x,u)}{u^{N-2}} = K \lim_{u\to\infty} u\left[\left(1 + \frac{\bar{u}}{u}\right)^{N-1} - 1\right] = K(N-1)\bar{u} \tag{4.5}$$

uniformly in B.

4.1. The case $3 \leqslant n \leqslant 5$

THEOREM 4.1. – *Let $3 \leqslant n \leqslant 5$ and let $K \in C^2(\overline{M})$ be strictly positive with a maximum at a point $P \in M - \partial M$. Suppose that the operator $\Delta + a$ is coercive on $\mathring{H}_1(M)$ with $a \in C^1(\overline{M})$. Then given a function $r \in C^1(\overline{M})$, $0 \leqslant r \not\equiv 0$, there exists a real number $\lambda^* > 0$ such that problem* (4.1) *has*:

(i) *no solution for $t > \lambda^*$;*
(ii) *just one solution for $t = \lambda^*$;*
(iii) *at least two distinct solutions for $0 < t < \lambda^*$.*

Proof. – The case $n = 3$ ($N = 6$). We deduce from (4.5) that

$$\lim_{u\to\infty} f(x,u)/u^3 = \infty \quad \text{uniformly in } x \in B. \tag{4.6}$$

Thus problem (4.3) has a solution by Theorem 3.2. And the theorem is proved in this case.

The cases $n = 4$ and 5 ($N = 4, 10/3$). It follows from (4.5) that

$$\lim_{u\to\infty} f(x,u)/u = \infty \quad \text{uniformly in } x \in B. \tag{4.7}$$

Consequently, we get a solution of problem (4.3) by using Theorem 3.1. So, the theorem is proved in this case too. □

4.2. The effect of linear term

THEOREM 4.2. – *Let $n = 6$ and let the operator $\Delta + a$ be coercive on $\mathring{H}_1(M)$ with $a \in C^1(\overline{M})$. Suppose that $K \in C^4(\overline{M})$ is strictly positive and achieves its maximum at a point $P \in M - \partial M$ such that*

$$5(a - S) + (\Delta K/K) \leqslant 0 \quad \textit{at } P.$$

Then given a function $r \in C^1(\overline{M})$, $0 \leqslant r \not\equiv 0$, there exists a real number $\lambda^* > 0$ such that problem (4.1) *has*:

(i) *no solution for $t > \lambda^*$;*
(ii) *exactly one solution for $t = \lambda^*$;*
(iii) *at last two distinct solutions for $0 < t < \lambda^*$.*

THEOREM 4.3. – *Let $n > 6$ and let the operator $\Delta + a$ be coercive on $\mathring{H}_1(M)$ with $a \in C^1(\overline{M})$. Suppose that $K \in C^4(\overline{M})$ is strictly positive* *and achieves its maximum at a point* $P \in M - \partial M$ *such that*

$$\frac{4(n-1)}{(n-2)(n-4)}(a-S) + \frac{\Delta K}{2K} < 0 \quad \textit{at } P.$$

Then given $r \in C^1(\overline{M})$, $0 \leqslant r \not\equiv 0$, *there exists a* $\lambda^* > 0$ *such that problem* (4.1) *has*:

(i) *no solution for* $t > \lambda^*$;
(ii) *exactly one solution for* $t = \lambda^*$;
(iii) *at least two distinct solutions for* $0 < t < \lambda^*$.

Here $S = (n-2)R/4(n-1)$, R being the scalar curvature of (M, g).

Proof of Theorems 4.2 *and* 4.3. –

(a) The case $n > 6$. Clearly, the conditions of Theorem 3.3 concerning Eq. (4.3) are well satisfied, thus Eq. (4.3) has a solution in this case, and Theorem 4.3 is proved.

(b) The case $n = 6$ $(N = 3)$. In this case, problem (4.3) is reduced to the following

$$\Delta u + \bar{a}u = Ku^2. \tag{4.8}$$

Here $\bar{a} = a - 2K\bar{u}$. It can be seen from (4.2) that the operator $\Delta + \bar{a}$ is coercive. Set $f(x, u) \equiv 0$, then hypothesis (H) is satisfied for problem (4.8). Moreover, we have

$$5(\bar{a} - S) + (\Delta K/K) < 5(a - S) + (\Delta K/K) \leqslant 0 \quad \text{at } P.$$

Thus (4.8) has a solution by Theorem 3.3, and Theorem 4.2 is proved.

4.3. Problems concerning the conformal Laplacian

Now consider the following problem

$$\begin{aligned} &\Delta u + Su = Ku^{N-1} + tr(x) && \text{in } M,\\ &u > 0 && \text{in } M,\\ &u = 0 && \text{on } \partial M, \end{aligned} \tag{4.9}$$

where $\Delta + S$ is the conformal Laplacian, $S = (n-2)R/4(n-1)$, R being the scalar curvature of (M, g).

THEOREM 4.4. – *Let* $n > 6$ *and let* $\Delta + S$ *be coercive. Suppose that* $K \in C^6(\overline{M})$ *be strictly positive on* $\overline{M}$*. Define*

$$\Omega_1 = \{Q \in M - \partial M \mid |W(Q)| \neq 0\},$$
$$\Omega_2 = \{Q \in M - \partial M \mid K(Q) = \max K\},$$
$$\Omega_3 = \{Q \in M \mid \Delta K(Q) = \Delta^2 K(Q) = 0\}.$$

If $\Omega_1 \cap \Omega_2 \cap \Omega_3 \neq \emptyset$*, then given a function* $r \in C^1(\overline{M})$*,* $0 \leqslant r \not\equiv 0$*, there exists a real number* $\lambda^* > 0$ *such that problem* (4.9) *has*:

(i) *no solution for* $t > \lambda^*$;
(ii) *exactly one solution for* $t = \lambda^*$;
(iii) *at least two distinct solutions for* $0 < t < \lambda^*$.

Proof. – By the expression (4.4) of f and inequality (4.2), it is easily checked that hypothesis (H) of Section 3 is well satisfied for problem (4.9). Since $\Omega_1 \cap \Omega_2 \cap \Omega_3 \neq \emptyset$, we get a solution of problem (4.9) by applying Theorem 3.4. Thus Theorem 4.4 is proved. □

THEOREM 4.5. – *Let* $n = 6$ *and let* $\Delta + S$ *be coercive. Suppose that* $K \in C^4(\overline{M})$ *is strictly positive on* $\overline{M}$*. If there exists a point* $P \in M - \partial M$ *where* K *is maximum and* $\Delta K(P) = 0$*, then given a function* $r \in C^1(\overline{M})$*,* $0 \leqslant r \not\equiv 0$*, there exists a real number* $\lambda^* > 0$ *such that problem* (4.9) *has*:

(i) *no solution for* $t > \lambda^*$;
(ii) *exactly one solution for* $t = \lambda^*$;
(iii) *at least two distinct solutions for* $0 < t < \lambda^*$.

Proof. – The conclusions of the theorem is proved with $a = S$ in Theorem 4.2. □

THEOREM 4.6. – *Let* (M, g) *be a locally conformally flat manifold of dimension* $n \geqslant 3$*. Suppose that the conformal Laplacian* $\Delta + a$ *is coercive and that* $K \in C^1(\overline{M})$*, with* $l = [n/2] + 1$*, be a strictly positive function on* $\overline{M}$*. Define*

$$\Omega_1 = \{Q \in M - \partial M \mid K(Q) = \max K\},$$
$$\Omega_2 = \{Q \in M \mid \Delta^k K(Q) = 0, 1 \leqslant k \leqslant (n-2)/4\}.$$

If $\Omega_1 \cap \Omega_2 \neq \emptyset$ *then given a function* $r \in C^1(\overline{M})$*,* $0 \leqslant r \not\equiv 0$*, there exists a real number* $\lambda^* > 0$ *such that problem* (4.9) *has*:

(i) *no solution for* $t > \lambda^*$;
(ii) *exactly one solution for* $t = \lambda^*$;
(iii) *at least two distinct solutions for* $0 < t < \lambda^*$.

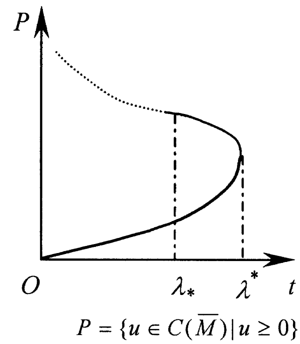

Fig. 1.

Proof. – It is obvious from the expression (4.4) of f and inequality (4.2) that conditions (H) in Section 3 concerning Eq. (4.9) are well satisfied. Furthermore, let $B \Subset M$ be a neighborhood of $P \in \Omega_1 \cap \Omega_2$, then we know by (4.5) that

$$\lim_{u\to\infty} \inf F(x,u)/u^{N-1} \geqslant C$$

uniformly in $x \in B$ ($C > 0$ is a constant). Since $\Omega_1 \cap \Omega_2 \neq \emptyset$, we get a solution of (4.9) by applying Theorem 3.5. This completes the proof of the theorem. □

As an immediate consequence, we have the following corollary.

COROLLARY 4.1. – *Let (M, g) be an $n \geqslant 3$ dimensional, locally conformally flat manifold, without boundary or with a sufficiently smooth boundary. Suppose that the conformal Laplacian $\Delta + a$ is coercive and that $K \equiv 1$. Then for every function $r \in C^1(\overline{M})$, $0 \leqslant r \not\equiv 0$, there exists a real number $\lambda^* > 0$ such that problem* (4.9) *has*

(i) *no solution for $t > \lambda^*$;*
(ii) *just one solution for $t = \lambda^*$;*
(iii) *at least two distinct solutions for $0 < t < \lambda^*$.*

It is the case if (M, g) is a bounded regular domain of $\mathbb{R}^n$ with the standard Euclidian metric.

Remark 4.1. – The solution structure of problem (4.1) and (4.9) given by Theorems 4.1–4.6 is roughly as indicated by the graph. The second solution $\bar{u}+u$ obtained through variational method is much greater than the minimal solution $\bar{u}=u(t)$. As t tends to zero, $\|\bar{u}(t)\|_{C(\overline{M})}$ tends to zero while $\|\nabla u\|_2^2$ is bounded from below by a positive constant C independent of t. We wonder if some L_p $(p>n/2)$ norm of the large solution is also bounded from above under appropriate conditions. If (M,g) is a starshaped domain in $\mathbb{R}^n$ and $K\equiv 1$, we know that $\|\nabla u\|_2^2$ will tend to infinity since problem (4.9) has no solution according to the Pohozaev identity.

REFERENCES

[1] Aubin T., Nonlinear Analysis on Manifolds, Monge–Ampere Equations, Grundlehern 252, Springer, 1982.

[2] Aubin T., Some Nonlinear Problems in Riemannian Geometry, Springer, 1998.

[3] Aubin T., Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, J. Math. Pures Appl. 55 (1976) 269–296.

[4] Aubin T., Espaces de Sobolev sur les variétés riemanniennes, Bull. Sci. Math. 100 (1976) 149–193.

[5] Aubin T., Problèmes isopérimétriques et espaces de Sobolev, J. Differential Geom. 11 (1976) 573–598.

[6] Aubin T., Hebey E., Courbure scalaire prescrite, Bull. Sci. Math. 115 (1991) 125–132.

[7] Aubin T., Bahri A., Méthodes de topologie algébrique pour le problème de la courbure scalaire prescrite, J. Math. Pures Appl. 76 (1997) 525–549.

[8] Aubin T., Bahri A., Une hypothèse topologique pour le problème de la courbure scalaire prescrite, J. Math. Pures Appl. 76 (1997) 843–850.

[9] Aubin T., Wang W., Fixed points of convex maps in ordered Banach spaces (to appear).

[10] Amann H., Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev. 18 (1976) 620–709.

[11] Ambrosetti A., Prodi G., On the inversion of some differential mapping with singularities between Banach spaces, Annali Mat. Pura Appl. 93 (1972) 231–247.

[12] Bahri A., Coron J.M., On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain, Comm. Pure Appl. Math. 41 (1988) 253–294.

[13] Berger M.S., Nonlinearity and Functional Analysis, Academic Press, 1997.

[14] Berestycki H., Le nombre de solutions de certains problèmes semi-linéaires elliptiques, J. Funct. Anal. 40 (1981) 1–29.

[15] Brezis H., Nirenberg L., Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure Appl. Math. 36 (1983) 437–477.

[16] Brezis H., Coron J.M., Multiple solutions of H-systems and Rellich's conjecture, Comm. Pure Appl. Math. 37 (1984) 149–198.

[17] Brezis H., Kato T., Remarks on the Schrödinger operator with singular complex potential, J. Math. Pures Appl. 58 (1979) 137–151.
[18] Escobar J.F., Positive solutions for some semilinear elliptic equations with critical Sobolev exponents, Comm. Pure Appl. Math. 40 (1987) 623–657.
[19] Escobar J.F., Schoen R., Conformal metrics with prescribed scalar curvature, Invent. Math. 86 (1986) 243–254.
[20] De Figueiedo D.G., On the superlinear Ambrosetti–Prodi problem, Nonlinear Anal. 8 (1984) 655–665.
[21] Gilbarg D., Trudinger N.S., Elliptic Partial Differential Equations of Second Order, 2nd ed., Springer-Verlag, 1983.
[22] Grandall M.G., Rabinowitz P., Some continuation and variational methods for positive solutions of nonlinear eigenvalue problems, Arch. Rat. Mech. Anal. 46 (1975) 81–95.
[23] Hebey E., Vaugon M., Meilleures constantes dans le théorème d'inclusion de Sobolev, Ann. de l'institut Henri Poincaré, Analyse non linéaire 13 (1996) 57–93.
[24] Hebey E., Vaugon M., Courbure scalaire prescrite pour des variétés non conformément difféomorphes à la sphère, C. R. Acad. Paris 316 (1993) 281–282.
[25] Hebey E., Vaugon M., Remarque sur le problème de Yamabe, J. Funct. Anal. 96 (1991) 31–37.
[26] Lions P.-L., Application de la méthode de concentration-compacité à l'existence de fonctions extrémales, C. R. Acad. Sci. Paris 296 (sèr I) (1983) 645–648.
[27] Lee J., Parker T., The Yamabe problem, Bull. Amer. Math. Soc. 17 (1987) 37–91.
[28] Pohozaev S., Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Soviet Math. Dokl. 6 (1965) 1408–1411.
[29] Sacks J., Uhlenbeck K., The existence of minimal 2-spheres, Ann. of Math. 113 (1981) 1–24.
[30] Schoen R., Conformal deformation of Riemannian metrics to constant scalar curvature, J. Differential Geom. 20 (1984) 479–495.
[31] Schoen R., Yau S.-T., Lectures on Differential Geometry, International Press, 1994.
[32] Trudinger N., Remarks concerning the conformal deformation of Riemannian structures on compact manifolds, Ann. Scuola Norm. Sup. Pissa 22 (1968) 265–274.
[33] Vaugon M., Equations différentielles non linéaires sur les variétés Riemanniennes compactes I, Bull. Sc. Math. 106 (1982) 351–367; II 107, 1983, 371–391.
[34] Yamabe H., On a deformation of Riemannian structures on compact manifolds, Osaka Math. J. 12 (1960) 21–37.
[35] Wang W., Problème d'Ambrosetti–Prodi avec exposant critique sur les variété riemanniennes compactes, Thèse de l'Université Pierre et Marrie Curie, 1998.